

Time-dependent Heston model.

G. S. Vasilev^{1,2}

¹*Department of Physics, Sofia University, James Bourchier 5 blvd, 1164 Sofia, Bulgaria*

²*CloudRisk Ltd*

(Dated: February 25, 2014)

This work presents an exact solution to the generalized Heston model, where the model parameters are assumed to have linear time dependence. The solution for the model is expressed in terms of confluent hypergeometric functions.

PACS numbers:

I. INTRODUCTION

The paper presents generalization of Heston's (1993) [1] stochastic volatility model based on time-dependent model parameters. The Heston model is one of the most widely used stochastic volatility (SV) models today. The Heston model can be viewed as a stochastic volatility generalization of the Black and Scholes (1973) [2] model and includes it as a special case. It is well known that the implied volatility smile used with Black-Scholes formula tends to systematically misprice out-of-the-money and in-the-money options if the volatility implied from the at-the-money option has been used. According to this effect various SV option pricing models (Hull and White (1987) [3], Stein and Stein (1991) [4], Heston (1993) [1]) have been developed to capture the "smile" effect. The popularity and the attractiveness of the Heston model lies in the combination of its three main features: it does not allow negative volatility, it allows the correlation between asset returns and volatility and it has a closed-form pricing formula. However, to use Heston model for option pricing one needs to know the model structural parameters. This brings us to the calibration problem and in the case of Heston model it faces several difficulties. For example the data used for model calibration are observed at discrete times, but the model is built under a continuous-time framework.

Within the calibration problem lies the need of generalization of the Heston model. Generally speaking, because the prices from stochastic engines (in our case the Heston model) are not supported by market prices, as a result the model parameters have to be recalibrated every day to new market data. This solution to calibration issue, apart from time consuming is not consistent with an accurate description of the dynamics. This is the main motivation to consider the Heston model with time-dependent parameters.

II. HESTON'S STOCHASTIC VOLATILITY MODEL

In this section the overview of the Heston model will be presented. We will derive the pricing partial differential equation (PDE), which forms the basis of the derivation

of the characteristic function in the next section. The Heston model was first introduced in [1], where we have two stochastic differential equations, one for the underlying asset price $S(t)$ and one for the variance $V(t)$ of $\log S(t)$:

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu(t)dt + \sqrt{V(t)}dW_1 \\ dV(t) &= \kappa(\theta - V(t))dt + \eta\sqrt{V(t)}dW_2.\end{aligned}\quad (1)$$

Here $\kappa \geq 0, \theta \geq 0$ and $\eta > 0$ stands for speed of mean reversion, the mean level of variance and the volatility of the volatility, respectively. Furthermore, the Brownian motions W_1 and W_2 are assumed to be correlated with correlation coefficient ρ . From the Heston model definition given by Eq.(1), it is trivial observation that SDE for the variance can be recognized as a mean-reverting square root process - CIR process, a process originally proposed by Cox, Ingersoll & Ross (1985) [5] to model the spot interest rate. Using Ito's lemma and standard arbitrage arguments we arrive at Garman's pricing PDE for the Heston model which reads

$$\begin{aligned}S \frac{\partial P}{\partial S} - rP &= [\kappa(V - \theta) - \lambda V] \frac{\partial P}{\partial V}, \\ \frac{\partial P}{\partial t} + \frac{1}{2}V S^2 \frac{\partial^2 P}{\partial S^2} + \rho \eta S V \frac{\partial^2 P}{\partial S \partial V} + \frac{1}{2}V \eta^2 \frac{\partial^2 P}{\partial V^2} &+ \end{aligned}\quad (2)$$

where λ is the market price of volatility risk. For the reader convenience a detail derivation is given in Appendix A. We further simplify the pricing PDE given by Eq.(??) by defining the forward option price

$$C_u(x(t), V(t), t) = e^{r(T-t)} P(S(t), V(t), t)$$

in which

$$x(t) = \log \left[\frac{e^{r(T-t)} S(t)}{K} \right]$$

Finally, define $\tau = T - t$, then we obtain the so-called forward equation in the form

$$\begin{aligned} & -\frac{\partial C_u}{\partial t} + \frac{1}{2}V \left[\frac{\partial^2 C_u}{\partial x^2} - \frac{\partial C_u}{\partial x} \right] \quad (\mathfrak{A}) \\ \rho \eta V \frac{\partial^2 C_u}{\partial x \partial V} + \frac{1}{2}V \eta^2 \frac{\partial^2 C_u}{\partial V^2} &= \kappa(V - \theta) \frac{\partial C_u}{\partial V} \end{aligned}$$

A. Characteristic function of the Heston model

Prior presenting the results related to characteristic function of the Heston model, we will refer to some results for affine diffusion processes. Following the work of Duffie, Pan and Singleton (2000) [6], for affine diffusion processes the characteristic function of $x(T)$ reads

$$f(x, V, \tau, \omega) = \exp [A(\omega, \tau) + B(\omega, \tau) V + C(\omega, \tau) x] \quad (4)$$

where is assumed $V \equiv V(t)$ and $x \equiv x(t)$. Moreover, the characteristic function must satisfy the following initial condition

$$f(x, V, 0, \omega) = \exp [i\omega x(T)],$$

which implies that

$$A(\omega, 0) = 0; \quad B(\omega, 0) = 0; \quad C(\omega, 0) = i\omega \quad (5)$$

Substituting characteristic function given by Eq.(4) in forward pricing PDE Eq.(3), according to the initial conditions, can be shown that the characteristic function simplifies to

$$f(x, V, \tau, \omega) = \exp [A(\omega, \tau) + B(\omega, \tau) V + i\omega x] \quad (6)$$

The functions $A(\omega, \tau)$ and $B(\omega, \tau)$ satisfy the following system of ordinary differential equations ODE

$$\frac{dA}{d\tau} = aB, \quad A(\omega, 0) = 0 \quad (7a)$$

$$\frac{dB}{d\tau} = \alpha - \beta B + \gamma B^2, \quad B(\omega, 0) = 0, \quad (7b)$$

where

$$\begin{aligned} a &= \kappa\theta \\ \alpha &= -\frac{1}{2}(\omega^2 + i\omega) \\ \beta &= \kappa - \rho\eta i\omega \\ \gamma &= \frac{1}{2}\eta^2 \end{aligned} \quad (8)$$

and $\omega \in \mathbf{R}$.

For the reader's convenience, in Appendix B we will give the derivation of the ODE in Eq.(7).

III. HESTON'S STOCHASTIC VOLATILITY MODEL WITH TIME DEPENDENT MODEL PARAMETERS

Since

$$f(x, V, \tau, \omega) = \mathbf{E}^{\mathbf{Q}} \left[e^{i\omega x(T)} \right],$$

where \mathbf{Q} is some risk-neutral measure, the problem for exact analytic solution for the Heston model is cast to problem for the respective characteristic function or more

precisely solutions for the $A(\omega, \tau)$ and $B(\omega, \tau)$ factors for the characteristic function. There are several directions towards generalizations of the Heston model with time-dependent parameters. Mikhailov and Nogel (2003) [7] indicate that θ can be relaxed from the constrain to be constant. As can be seen from Eq.(7a) the parameter θ can be assumed time dependent and the exact solution of $A(\omega, \tau)$ is still possible. Mikhailov and Nogel indicate that other choices of time dependent parameters are still possible but the general solution of the ODE system Eq.(7) is restricted from the Riccati equation given in Eq.(7b). Other possible generalizations are related to time discrimination - piece-wise constant parameters or asymptotic solutions.

Our approach assumes that all model parameters have linear time dependence. I.e. κ, θ, η and ρ are linear in respect in time. Although this is a restrictive, we would like to note that locally, arbitrary time dependence is reduced to linear. Also such generalization is the simplest non-trivial, where all model parameters are time-dependent.

We should stress that our main concern is finding a solution of the Eq.(7b), because the solution of the $A(\omega, \tau)$ is simply given by integration with integrand the desired solution of $B(\omega, \tau)$ times a .

A. Constant volatility of the volatility parameter

We should stress that even within this framework, assuming linear time-dependence for all model parameters, the solution is complicated. With this reasoning, we first will consider constant η and linear κ, θ and ρ .

$$\begin{aligned} \eta &= \text{const} \\ \kappa &= \kappa_1\tau + \kappa_2 \\ \theta &= \theta_1\tau + \theta_2 \\ \rho &= \rho_1\tau + \rho_2 \end{aligned} \quad (9)$$

Having defined model parameters the next step is looking for solutions. Because the analytic theory for second order linear ODE with time-dependent coefficients is well studied, our first step is to cast the Riccati equation in Eq.(7b) to linear second order ODE. In Eq.(7b), after the substitution

$$B = -\frac{\dot{D}}{\gamma D} \quad (10)$$

where $\dot{D} = \frac{dD}{d\tau}$, we obtain

$$\frac{d^2 D}{d\tau^2} + \beta \frac{dD}{d\tau} + \alpha\gamma D = 0 \quad (11)$$

According to the substitution Eq.(10) and the initial conditions Eq.(5) the new initial condition for $D(\tau)$ reads

$$\dot{D}(0) = 0 \quad (12a)$$

Note that we should have two initial conditions as we intend to solve second order ODE, but because the prime equation is Eq.(7b), which is first order, the initial condition in Eq.(12a) is sufficient. With the notation used we obtain the following equation to solve

$$\frac{d^2 D}{d\tau^2} + (g_1 \tau + g_2) \frac{dD}{d\tau} + \alpha \gamma D = 0, \quad (13)$$

where

$$\begin{aligned} g_1 &= \kappa_1 - i\rho_1 \eta \omega \\ g_2 &= \kappa_2 - i\rho_2 \eta \omega \end{aligned}$$

Eq.(13) can be transformed to confluent hypergeometric equation via the substitution

$$z = -\frac{(g_1 \tau + g_2)^2}{2g_1} \quad (14)$$

After simple algebra we obtain

$$z \frac{d^2 D}{dz^2} + \left(\frac{1}{2} - z\right) \frac{dD}{dz} - \frac{\alpha \gamma}{2g_1} D = 0 \quad (15)$$

If we compare the equation given Eq.(15) with the standard form of the confluent hypergeometric equation, which reads

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0 \quad (16)$$

we immediately can write the desired solution. Basic reference formulas about confluent hypergeometric equation and its solutions are given in Appendix C. The solution for Eq.(15) is

$$D(\tau) = A_1 M \left[\frac{\alpha \gamma}{2g_1}, \frac{1}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right] + \quad (17)$$

$$A_2 U \left[\frac{\alpha \gamma}{2g_1}, \frac{1}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right]. \quad (18)$$

Here $M \left[\frac{\alpha \gamma}{2g_1}, \frac{1}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right]$ and $U \left[\frac{\alpha \gamma}{2g_1}, \frac{1}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right]$ are standard notations for the Kummer functions [9] representing the two linear independent solutions of Eq.(16). From Appendix C, because $b = \frac{1}{2}$, second solution $U(a, b, z)$ of Eq.(16) is given by convergent series for all values if a and b . Next step towards completing the solution for the time-dependent Heston model is to obtain the integration constants A_1 and A_2 from the initial condition.

Using Eq.(36) and Eq.(37) from the solution given in Eq.(17) and the initial condition in Eq.(12a), we obtain

$$\left. \frac{d}{d\tau} D(\tau) \right|_{\tau=0} = A_1 \frac{\alpha \gamma}{g_1} M \left[\frac{\alpha \gamma}{2g_1} + 1, \frac{3}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right] (-g_1 \tau - g_2) + A_2 \left(-\frac{\alpha \gamma}{2g_1} \right) U \left[\frac{\alpha \gamma}{2g_1} + 1, \frac{3}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right] (-g_1 \tau - g_2) \Big|_{\tau=0}$$

From the above Eq.(??) we obtain the functional relation between A_1 and A_2

$$A_1 = \frac{A_2}{2} \frac{U \left[(\alpha \gamma + 2g_1)/2g_1, 3/2, -(g_2)^2/2g_1 \right]}{M \left[(\alpha \gamma + 2g_1)/2g_1, 3/2, -(g_2)^2/2g_1 \right]} = \frac{A_2}{2} F \quad (19)$$

To shorten the notations we have used F to denote the

U/M fraction.

As we have mentioned due to the homogenous transformation given in Eq.(10) the functional relation between A_1 and A_2 in Eq.(19) is sufficient to determine the function $B(\omega, \tau)$ of the characteristic function for the Heston model

$$B(\omega, \tau) = \frac{\alpha}{g_1} (g_1 \tau + g_2) \left\{ \frac{U \left[\frac{\alpha \gamma}{2g_1} + 1, \frac{3}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right] - F M \left[\frac{\alpha \gamma}{2g_1} + 1, \frac{3}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right]}{F M \left[\frac{\alpha \gamma}{2g_1}, \frac{1}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right] + U \left[\frac{\alpha \gamma}{2g_1}, \frac{1}{2}, -\frac{(g_1 \tau + g_2)^2}{2g_1} \right]} \right\} \quad (20)$$

B. Linear in time volatility of the volatility parameter

Let us derive the solution for the general case of linear time dependence. The model parameters have the

following form

$$\begin{aligned}
\eta &= \eta_1\tau + \eta_2 \\
\kappa &= \kappa_1\tau + \kappa_2 \\
\theta &= \theta_1\tau + \theta_2 \\
\rho &= \rho_1\tau + \rho_2
\end{aligned} \tag{21}$$

Again one can apply the transformation given by Eq.(10) to the Eq.(7b). After some algebra we obtain the following equation

$$\frac{d^2 D}{d\tau^2} + (h_1\tau^2 + h_2\tau + h_3)\frac{dD}{d\tau} + (k_1\tau + k_2)^2 D = 0, \tag{22}$$

where the coefficients h_1, h_2, h_3, k_1 and k_2 are given as follows

$$\begin{aligned}
h_1 &= -i\omega\rho_1\eta_1 \\
h_2 &= \kappa_1 - (\rho_1\eta_2 + \rho_2\eta_1)i\omega \\
h_3 &= \kappa_2 - \rho_2\eta_2 i\omega \\
k_1 &= \frac{\eta_1\sqrt{\alpha}}{\sqrt{2}} \\
k_2 &= \frac{\eta_2\sqrt{\alpha}}{\sqrt{2}}
\end{aligned}$$

The solution of the Eq.(22) can be cast to the tree-confluent Heun equation [10] given by Eq.(40).

$$D(\tau) = A_1 \exp(f) \text{HeunT}(\alpha, \beta, \gamma, z) + \tag{23}$$

$$A_2 \exp\left[-\frac{\tau(k_1)^2}{h_1}\right] \text{HeunT}(\alpha, \beta, \gamma, z) \tag{24}$$

where we have used the following notations

$$f = -\frac{\tau \left[2(h_1\tau)^2 + 3h_1h_2\tau + 6h_1h_3 - 6(k_1)^2 \right]}{6h_1} \tag{25}$$

$$\alpha = \frac{3^{2/3}}{2(h_1)^{8/3}} \left[(h_2k_1)^2 + 2(k_2k_1)^2 - 2h_1h_2k_2k_1 \right] \tag{26}$$

$$-2(k_1)^2 h_1h_3 + 2(k_1)^4 \tag{27}$$

$$\beta = -\frac{3 \left[(k_1)^2 h_2 + (k_1)^2 - 2h_1k_2k_1 \right]}{(h_1)^2} \tag{28}$$

$$\gamma = \frac{3^{1/3} \left[4h_1h_3 - (h_2)^2 - 8(k_1)^2 \right]}{4(h_1)^{4/3}} \tag{29}$$

Again for reader convenience basic definitions and formulas for Heun equations are given in Appendix D.

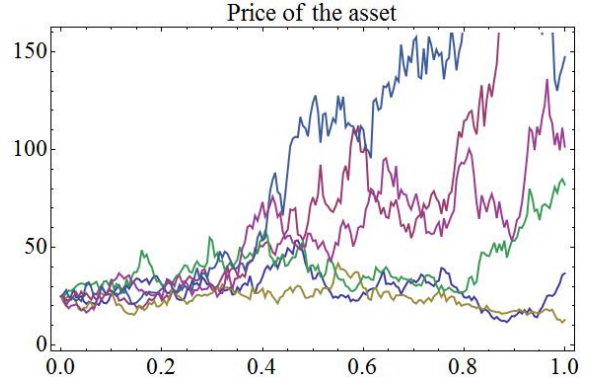


FIG. 1: Asset price govern by the Heston stochastic proces .

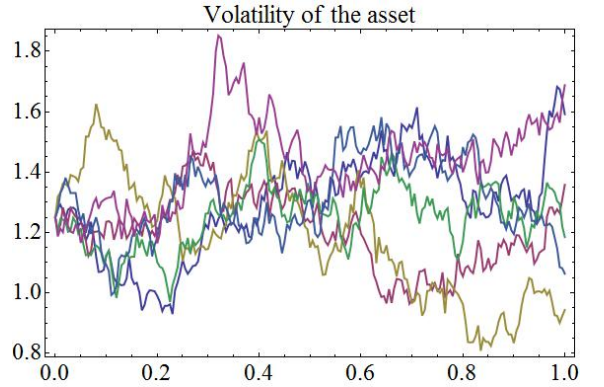


FIG. 2: Volatility related to Heston stochastic procec.

IV. CONCLUSIONS

The Heston model is among the most popular stochastic volatility models due to its analytical tractability. Nevertheless, the complete use of the Heston model is still challenging because it has a closed formula only when the parameters are constant or piecewise constant or within the asymptotic limits. The aim of this research is to fill this theoretical gap and to enrich the usability of this very popular model in respect to calibration issues.

There are several concluding remarks that should be also noted:

A similar to the presented above approach is applicable to the SVJ models and more precisely to the Bates model. This problem will be addressed in more details in forthcoming research.

Within the presented work we have assumed that all model parameters obey linear time dependence. This assumption have been made for consistency, but the more general assumption allow arbitrary time dependence for the θ Heston parameter.

Acknowledgments

This work has been supported by the project QUANT-NET - European Reintegration Grant (ERG) - PERG07-GA-2010-268432.

V. APPENDIX A

This appendix summarize a key formulas and their derivation for the Heston model. Using Itô's lemma for

a function $V(S_1, S_2, t)$, which is twice differentiable with respect to S_1 and S_2 , and once with respect to t , where

$$\begin{aligned} dS_1 &= a_1(S_1, S_2, t)dt + b_1(S_1, S_2, t)dW_1 \\ dS_2 &= a_2(S_1, S_2, t)dt + b_2(S_1, S_2, t)dW_2 \end{aligned}$$

and W_1 and W_2 are correlated with correlation coefficient ρ , for the differential of $V(S_1, S_2, t)$ due to Itô's lemma we obtain

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S_1}dS_1 + \frac{\partial V}{\partial S_2}dS_2 + \frac{1}{2}b_1^2 \frac{\partial^2 V}{\partial S_1^2}dt + \frac{1}{2}b_2^2 \frac{\partial^2 V}{\partial S_2^2}dt + \rho b_1 b_2 \frac{\partial^2 V}{\partial S_1 \partial S_2}$$

It is standard to use the above equation Eq.(??) for a function assuming a self-financing portfolio with value consisting of an option with value $P(S, V, t)$, $-\Delta$ units of the

underlying asset S and, in order to hedge the risk associated with the random volatility, $-\Delta_1$ units of another

option with value $P_1(S, V, t)$. Hence we get

$$\Pi = P - \Delta S - \Delta_1 P_1$$

Using again Itô's lemma after some algebra we obtain

$$\begin{aligned} d\Pi &= \left(\frac{\partial P}{\partial t} + \frac{1}{2}VS^2 \frac{\partial^2 P}{\partial S^2} + \rho\eta SV \frac{\partial^2 P}{\partial S \partial V} + \frac{1}{2}V\eta^2 \frac{\partial^2 P}{\partial V^2} \right) dt - \Delta_1 \left(\frac{\partial P_1}{\partial t} + \frac{1}{2}VS^2 \frac{\partial^2 P_1}{\partial S^2} + \rho\eta SV \frac{\partial^2 P_1}{\partial S \partial V} + \frac{1}{2}V\eta^2 \frac{\partial^2 P_1}{\partial V^2} \right) dt \\ &\quad + \left(\frac{\partial P}{\partial S} - \Delta_1 \frac{\partial P_1}{\partial S} - \Delta \right) dS + \left(\frac{\partial P}{\partial V} - \Delta_1 \frac{\partial P_1}{\partial V} \right) dV \end{aligned}$$

Following the standard approach if one assumes

$$\begin{aligned} \frac{\partial P}{\partial S} - \Delta_1 \frac{\partial P_1}{\partial S} - \Delta &= 0 \\ \frac{\partial P}{\partial V} - \Delta_1 \frac{\partial P_1}{\partial V} &= 0 \end{aligned} \quad (30a)$$

one can get a risk-free portfolio. The standard approach requires to eliminate arbitrage opportunities and hence the return of this risk-free portfolio must equal the (deterministic) risk-free rate of return r :

$$\begin{aligned} d\Pi &= \left(\frac{\partial P}{\partial t} + \frac{1}{2}VS^2 \frac{\partial^2 P}{\partial S^2} + \rho\eta SV \frac{\partial^2 P}{\partial S \partial V} + \frac{1}{2}V\eta^2 \frac{\partial^2 P}{\partial V^2} \right) dt - \Delta_1 \left(\frac{\partial P_1}{\partial t} + \frac{1}{2}VS^2 \frac{\partial^2 P_1}{\partial S^2} + \rho\eta SV \frac{\partial^2 P_1}{\partial S \partial V} + \frac{1}{2}V\eta^2 \frac{\partial^2 P_1}{\partial V^2} \right) dt \\ &= r\Pi dt = r(P - \Delta S - \Delta_1 P_1) dt \end{aligned}$$

Using the conditions in Eq.(30a) the above equation for the risk-free portfolio Eq.(??) become

$$\begin{aligned} &\left(\frac{\partial P}{\partial t} + \frac{1}{2}VS^2 \frac{\partial^2 P}{\partial S^2} + \rho\eta SV \frac{\partial^2 P}{\partial S \partial V} + \frac{1}{2}V\eta^2 \frac{\partial^2 P}{\partial V^2} + rS \frac{\partial P}{\partial S} - rV \right) / \frac{\partial P}{\partial V} \\ &= \left(\frac{\partial P_1}{\partial t} + \frac{1}{2}VS^2 \frac{\partial^2 P_1}{\partial S^2} + \rho\eta SV \frac{\partial^2 P_1}{\partial S \partial V} + \frac{1}{2}V\eta^2 \frac{\partial^2 P_1}{\partial V^2} + rS \frac{\partial P_1}{\partial S} - rV \right) / \frac{\partial P_1}{\partial V} \end{aligned}$$

Having obtained this equation one can conclude that both left and right-hand sides should be equal to some function g that only depends on the independent variables S, V and t . If one define the function g to have the following form $g = \kappa(V - \theta) - \lambda V$ a special case of a so-called affine diffusion process is considered. For such class of processes, the pricing PDE is tractable analytically.

In the considered case the Garman's pricing PDE for the Heston model is given by Eq.(??)

VI. APPENDIX B

In this appendix some results regarding the work of Duffie, Pan and Singleton (2000) [6], will be summarized.

$$-f \left(\frac{\partial A}{\partial \tau} + \frac{\partial B}{\partial \tau} V \right) + \frac{1}{2} V f(-\omega^2 - i\omega) + \rho \eta V B f i\omega + \frac{1}{2} \eta^2 V B^2 f - \kappa(V - \theta) B f = 0 \quad (31)$$

If one use the notations given in Eq.(8), than Eq.(??) simplifies to

$$-\frac{\partial A}{\partial \tau} + aB + V \left(\frac{\partial B}{\partial \tau} + \alpha - \beta B + \gamma B^2 \right) = 0$$

The equation above is a first order polynomial in V . In order for this equation to hold, both coefficients must vanish, i.e. we obtain that $A(\omega, \tau)$ and $B(\omega, \tau)$ satisfy the system of ordinary differential equations ODE given in Eq.(7).

VII. APPENDIX C

This appendix is written for the readers convenience and contains basic definitions and formulas for the confluent hypergeometric function.

Standard form of the confluent hypergeometric equation reads

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0 \quad (32)$$

This equation has a regular singularity at $z = 0$ and an irregular singularity at $z = \infty$. There are eight linearly independent solutions, but the standard two are so-called Kummer's Functions

$$M(a, b, z) = 1 + \frac{az}{b} + \frac{(a)_2 z^2}{(b)_2 2!} + \dots + \frac{(a)_n z^n}{(b)_n n!} + \dots \quad (33)$$

where

$$(a)_n = a(a+1)(a+2)\dots(a+n-1), \quad (a)_0 = 1$$

For affine diffusion processes the characteristic function of $x(T)$ can be written as given in Eq.(6)

$$f(x, V, \tau, \omega) = \exp [A(\omega, \tau) + B(\omega, \tau) V + i\omega x]$$

The characteristic function $f(x, V, \tau, \omega)$ satisfies the forward equation given in Eq.(3). Substituting $f(x, V, \tau, \omega)$ given in Eq.(6) into Eq.(3) yields

and the second function is given by

$$U(a, b, z) = \frac{\pi}{\sin \pi b} \left[\frac{M(a, b, z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right] \quad (34)$$

Using the $M(a, b, z)$ and $U(a, b, z)$ the complete solution of the confluent hypergeometric equation Eq.(32) reads

$$w(z) = AM(a, b, z) + BU(a, b, z) \quad (35)$$

where A and B are arbitrary constants and $b \neq -n$. Within the related literature are used alternative notations for $M(a, b, z)$ and $U(a, b, z)$, which respectively reads ${}_1F_1(a; b; z)$ and $z^{-a} {}_2F_0(a, 1+a-b; ; -1/z)$. The derivative of the two independent solutions with respect to the independent variable z are given by

$$\frac{d}{dz} M(a, b, z) = \frac{a}{b} M(a+1, b+1, z) \quad (36)$$

and

$$\frac{d}{dz} U(a, b, z) = -aU(a+1, b+1, z) \quad (37)$$

VIII. APPENDIX D

The appendix contains basic definitions and formulas for the Heun equation and more precisely for the triconfluent Heun equation, which is one of the confluent forms

of Heun's equation. Heun's equation

$$\frac{d^2w}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a} \right) \frac{dw}{dz} + \frac{\alpha\beta z}{z(z-1)(z-a)} w = 0 \quad (38)$$

has regular singularities at $0, 1, a$ and ∞ . Generally speaking confluent forms of Heun's differential equation arise when two or more of the regular singularities merge to form an irregular singularity. This process is analogous to

the derivation of the confluent hypergeometric equation from the hypergeometric equation. There are four confluent standard forms and the triconfluent Heun Equation given by

$$\frac{d^2w}{dz^2} + (\gamma + z)z \frac{dw}{dz} + (\alpha z + q)w = 0 \quad (39)$$

This equation has one singularity, which is an irregular singularity of rank 3 at $z = \infty$. Within the literature one can find other canonical forms of the triconfluent Heun Equation. Following Slavyanov, S.Y., and Lay, W the

$\text{HeunT}(\alpha, \beta, \gamma, z)$ function is a local solution to Heun's Triconfluent equation,

$$\frac{d^2w}{dz^2} - (3z^2 - a) \frac{dw}{dz} - ((3 - b)z - a)w = 0 \quad (40)$$

computed as a standard power series expansion around the origin, which is a regular point. Because the single singularity is located at $z = \infty$, this series converges in the whole complex plane.

-
- [1] Heston S., A closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies* **6**, 327-343, (1993).
 - [2] Black, F. and Scholes, M., The pricing of options and corporate liabilities, *Journal of Political Economy* **81**, 637-659, (1973).
 - [3] Hull, J. and White, A., The pricing of options on assets with stochastic volatilities. *Journal of Finance*, **42**, 281-300, (1987).
 - [4] Stein, E. and Stein, J., Stock price distributions with stochastic volatility: an analytic approach. *Review of Financial Studies*, **4**, 727-752, (1991).
 - [5] Cox, J., Ingersoll, J. and Ross, S., A theory of the term structure of interest rates, *Econometrica* **53**, 385-408, (1985).
 - [6] Duffie, D., Pan, J. and Singleton, K., Transform analysis and asset pricing for affine jump diffusions, *Econometrica* **68**, 1343-1376, (2000).
 - [7] Mikhailov, S. and Nogel, U., Heston's Stochastic Volatility Model: Implementation, Calibration and Some Extensions, *The Best of Wilmott Vol 1*, (2003).
 - [8] Gatheral, J., *The Volatility Surface: A Practitioner's Guide*, (Wiley Finance (Book 357), 2006).
 - [9] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1964).
 - [10] Ronveaux, A. ed., *Heun's Differential Equations*, (Oxford University Press, 1995).